Math 131 B, Lecture 2
Analysis

Sample Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1. *10pts.*

Match each general statement on the first list below with its consequence on the real line on the second list.

List One: General Theorems

- (A) If $S \subset M$ is a dense subset of a metric space M, any continuous function $f : M \to T$ is determined by its values on S.
- (B) If S is compact, any infinite subset K of S has a limit point in S.
- (C) If p is a limit point of a set S, every neighbourhood of p contains infinitely many points of S.
- (D) The image of a connected set under a continuous map is connected.
- (E) The image of a compact set under a continuous map is compact.

List Two: Consequences in $\mathbb{R}$.

- (1) Let $f : [a, b] \to \mathbb{R}$ be continuous. If $x = \sup\{f(x) : x \in [a, b]\}$ and $y = \inf\{f(x) : x \in [a, b]\}$. Then there is $c, d \in [a, b]$ such that $f(c) = x$ and $f(d) = y$.
- (2) Every bounded sequence of real numbers has a convergent subsequence.
- (3) Let $f : [a, b] \to \mathbb{R}$ be continuous. If x is a real number which is between $f(a)$ and $f(b)$, there is some $c \in [a, b]$ such that $f(c) = x$.
- (4) If two continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ agree on all rational numbers, then they are the same function.
- (5) A real number y is a subsequential limit of a sequence $\{x_n\}$ if and only if $\{n : x_n \in (y - \epsilon, y + \epsilon)\}$ is infinite for all $\epsilon > 0$.

Solution: A-4, B-2, C-5, D-3, E-1

Problem 2.

(a) [5pts.] Give a definition of a connected metric space M.

Solution: We say that M is connected if there do not exist disjoint nonempty open sets A and B in M such that $M = A \cup B$.

(b) [5pts.] Let $S \subset M$ be a subset of a metric space. Prove or disprove: If $\overline{S}$ is connected, S is connected.

Solution: This is false. Consider $\mathbb{Q}$, which we saw in class is totally disconnected. The closure $\overline{\mathbb{Q}} = \mathbb{R}$ is connected.

Problem 3.

(a) [5pts.] Let $\{f_n\}$ be a sequence of functions $f_n : S \to T$. What does it mean for f_n to converge uniformly to a function $f : S \to T$?

Solution: We say that $f_n \to f$ uniformly if for every $\epsilon > 0$ there is some N such that for $n > N$ and all $x \in S$, we have $d_T(f_n(x), f(x)) < \epsilon$.

(b) [5pts.] Prove that if $f_n \to f$ uniformly and each f_n is integrable, f is integrable.

Solution: Let $\epsilon < 0$. Choose N such that $|f_n(x) - f(x)| < \epsilon$ for $n \geq N$. In particular, $f_n(x) - \epsilon < f(x) < f_n(x) + \epsilon$, and therefore $L(f_n - \epsilon) \leq L(f) \leq U(f) \leq U(f_n + \epsilon)$. Now $L(f_n - \epsilon) = L(f_n) - \epsilon(b - a) = U(f_n) - \epsilon(b - a)$, and $U(f_n + \epsilon) = U(f_n) + \epsilon(b - a)$, so we conclude that $U(f) - L(f) \leq 2\epsilon(b - a)$. Since ϵ was arbitrary, $U(f) = L(f)$ and f is integrable.

Problem 4.

(a) [5pts.] State the Weierstrass M-test.

Solution: If $\{f_n\}$ is a sequence of real-valued functions, and for each n we have some $M_n > 0$ such that $|f_n(x)| < M_n$ for all $x \in S$, and moreover the series $\sum M_n$ converges, then f_n converges uniformly on S.

(b) [5pts.] Prove that the function $f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n(1+n^2x^2)}$ is continuous. What is its antiderivative?

Solution: Notice that $|\frac{1}{2^n(1+n^2x^2)}| < \frac{1}{2^n}$, and $\sum \frac{1}{2^n}$ converges, so by the Weierstrass M-test, this series of functions converges uniformly. Since each term is continuous and uniform convergence preserves continuity, the sum is continuous. Finally, the antiderivative of the series is the sum of the antiderivatives of each term, to wit $\int f(x) = C + \sum_{n=0}^{\infty} \frac{\tan^{-1}(nx)}{n2^n}$.

Problem 5.

(a) [5pts.] Define the Cauchy product of two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$.

Solution: The Cauchy product is the power series $\sum_{n=0}^{\infty} c_n x^n$, where $c_n = \sum_{k=0}^{n} a_n b_{n-k}$.

(b) [5pts.] Let $c_n = \sum_{k=1}^{n} \frac{k}{n+1-k}$. What function is represented by the power series $\sum_{n=0}^{\infty} c_n x^n$ on $(-1, 1)$?

Solution: This power series is the Cauchy product of $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$ and $\sum_{n=1}^{\infty} \frac{x^n}{n} = \ln(1-x)$, so on $(-1, 1)$ it converges to $f(x) = \frac{\ln(1-x)}{(1-x)^2}$.